MAT2R3 TUTORIAL 8

Review:

(1) *The matrix of a linear transformation*

In finite dimensions, every linear transformation is just matrix multiplication. Lets first look at this for the vector space $\mathbb{R}^n$.

Let $B = \{e_1, e_2, \ldots, e_n\}$ be the standard basis (i.e. e_k has a 1 in the k'th entry and zeros everywhere else). Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and consider the matrix

$$A_T = \big(T(e_1) \quad | \quad T(e_2) \quad | \quad \ldots \quad |T(e_n)\big),$$

where $T(e_k)$ is the $k'th$ column of the matrix.

Then for any $x \in \mathbb{R}^n$ we can write x as a linear combination of the e_k, that is $x = a_1e_1 + \ldots + a_ne_n$. Then we see that $T(x) = a_1T(e_1) + \ldots + a_nT(e_n)$, since T is linear. Then if we compute the matrix product A_Tx, we see that

$$A_Tx = A_T(a_1e_1 + \ldots + a_ne_n) = A_Ta_1e_1 + \ldots + A_Ta_ne_n = a_1A_Te_1 + \ldots a_nA_Te_n.$$

If we think for a moment about how matrix multiplication is defined, we see that A_Te_k is the k'th column of the matrix A_T which is $T(e_k)$. So we have

$$A_Tx = a_1T(e_1) + \ldots + a_nT(e_n) = T(x).$$

Lets look at an example.

Example:

Let $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$. Then we have $A_T = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then for any $\begin{pmatrix} x \\ y \end{pmatrix}$ we have

$$A_T\begin{pmatrix} x \\ y \end{pmatrix} = xA_T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yA_T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}.$$

Now that we have T represented as a matrix, we can make calculations like determining its kernel and range and their dimensions.

(2) *The matrix of a general linear transformation*

What if the vector space we are working with is not $\mathbb{R}^n$? We can do that same thing!

Suppose that $T\colon V \to W$ is a linear transformation between finite dimensional vector spaces. Fix (ordered) bases $B = \{v_1, \ldots, v_n\}$ for V and $B' = \{w_1, \ldots, w_m\}$ for W. Now that we have made this choice, we can think of V and W as being $\mathbb{R}^n$ and $\mathbb{R}^m$.

More precisely, for any $v \in V$, we can write $v = a_1v_1 + \ldots + a_nv_n$. This shows that all we need to specify v is a list of n numbers. We identify v with the column vector $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$. This map $v \mapsto \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is an isomorphism (you should check this if Bradd did not show you the proof). We also have a similar identification for W and $\mathbb{R}^m$, if $w = b_1w_1 + \ldots + b_mw_m$, then we identify w with $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. We write $[v]_B$ for $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and similarly $[w]_{B'}$ for $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

Now we form the matrix

$$[T]_B^{B'} = \left([T(v_1)]_{B'} \quad | \ldots | \quad [T(v_n)]_{B'} \right).$$

Then we have $[T]_B^{B'}[v]_B = [T(v)]_{B'}$.

Lets look at an example.

Example:

Let $T\colon M_2 \to M_2$ be given by $T(A) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} A$. How do we represent T as a matrix transformation? First we need to choose bases for the domain, M_2, and the target, M_2. There is a very nice basis for M_2, so we might as well take this to be the basis for both the domain and the range.

Take $B = B' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

Now we calculate the columns of our matrix,

- $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$
- $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$
- $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

- $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

Then identifying these with columns, we have

$$[T_B^{B'}] = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now that we have represented T as a matrix, we can make the same calculations that we would for matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, like calculating the kernel and range and their dimensions.

Problem 1

Verify that $[T]_B^{B'}[A]_B = [T(A)]_{B'}$ for the example above. What is the dimension of the kernel and the range of this transformation? Is it one-to-one? onto? an isomorphism?

Problem 2

Let $T\colon P_1 \to \mathbb{R}^2$ be given by $T(a+bx) = \begin{pmatrix} a+b \\ b \end{pmatrix}$. Let $B = \{1, x\}$ and $B' = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$. Calculate the matrix $[T]_B^{B'}$. Is T one-to-one? onto? an isomorphism?